Exam Complex Analysis January 31, 2012

The exam consists of 5 problems. You should **write clearly, argue clearly, and motivate your answers**. Points available can be found below.

1. Consider the complex function $f(z) = x^3+3xy^2-3x+i(y^3+3x^2y-3y)$ for $z = x+iy$. Use the Cauchy-Riemann equations to show the following statements:

a. $f(z)$ is differentiable on both the coordinate axes $x = 0$ and $y = 0$.

b. $f(z)$ is nowhere analytic.

2. Consider the function $g(z) = \dfrac{e^z}{z^2-1}$.

a. Let Γ_1 be any simple closed positively oriented contour that lies entirely in the disc $|z| \leq \frac{1}{2}$. Determine $\int_{\Gamma_1} g(z)dz$.

b. Let Γ_2 be the circle $|z| = 2$, traversed once in counterclockwise direction. Compute $\int_{\Gamma_2} g(z)dz$.

3. Let $f(z)$ be the function given by $f(z) = z^2 \cos \dfrac{1}{z}$.

a. Determine the Laurent series for $f(z)$ in $|z| > 0$.

b. Compute the residue of $f(z)$ at $z = 0$.

c. What kind of singularity does $f(z)$ have at $z = 0$?

4. Consider the complex function $f(z) = \dfrac{ze^{iz}}{\sin z}$.

a. Determine the poles of $f(z)$. Also determine their orders.

b. What can one say about the point $z = 0$?

c. Show that $f(z)$ can be represented by a power series around $z = 0$. Determine its radius of convergence. (hint: you do not need to compute the series!)

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5. Rouché's theorem is a very powerful result to obtain information about the zeros of analytic functions.

$\frac{1}{2}$ **a.** Give a precise formulation of Rouché's theorem.

8 **b.** Consider the function $g(z) = 6z^4 + z^3 - 2z^2 + z - 1$. Determine the number of roots of the equation $g(z) = 0$ in the disc $|z| < 1$.

8 **c.** Give an example to show that the conclusion of Rouché's theorem may be false if the strict inequality $|h(z)) < |f(z)|$ is replaced by $|h(z)) \leq |f(z)|$ for z on the contour C.

Points:

Problem 1: 16 8
Problem 2: 18 18
Problem 3: 16 8 of 4.
Problem 4: 24 12
Problem 5: 16 12
10 points for free 10 +

3,4
3 +

6,4.